

## ON MULTIPARTITE HAJNAL-SZEMERÉDI THEOREMS

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ABSTRACT. Let  $G$  be a  $k$ -partite graph with  $n$  vertices in parts such that each vertex is adjacent to at least  $\delta^*(G)$  vertices in each of the other parts. Magyar and Martin [20] proved that for  $k = 3$ , if  $\delta^*(G) \geq \frac{2}{3}n + 1$  and  $n$  is sufficiently large, then  $G$  contains a  $K_3$ -factor (a spanning subgraph consisting of  $n$  vertex-disjoint copies of  $K_3$ ). Martin and Szemerédi [21] proved that  $G$  contains a  $K_4$ -factor when  $\delta^*(G) \geq \frac{3}{4}n$  and  $n$  is sufficiently large. Both results were proved using the Regularity Lemma. In this paper we give a proof of these two results by the absorbing method. Our absorbing lemma actually works for all  $k \geq 3$  and may be utilized to prove a general and tight multipartite Hajnal-Szemerédi theorem.

## 1. INTRODUCTION

Let  $H$  be a graph on  $h$  vertices, and let  $G$  be a graph on  $n$  vertices. *Packing* (or *tiling*) problems in extremal graph theory are investigations of conditions under which  $G$  must contain many vertex disjoint copies of  $H$  (as subgraphs), where minimum degree conditions are studied the most. An  $H$ -matching of  $G$  is a subgraph of  $G$  which consists of vertex-disjoint copies of  $H$ . A *perfect  $H$ -matching*, or  *$H$ -factor*, of  $G$  is an  $H$ -matching consisting of  $\lfloor n/h \rfloor$  copies of  $H$ . Let  $K_k$  denote the complete graph on  $k$  vertices. The celebrated theorem of Hajnal and Szemerédi [6] says that every  $n$ -vertex graph  $G$  with  $\delta(G) \geq (k-1)n/k$  contains a  $K_k$ -factor (see [11] for another proof).

Using the Regularity Lemma of Szemerédi [25], researchers have generalized this theorem for packing arbitrary  $H$  [1, 15, 24, 16]. Results and methods for packing problems can be found in the survey of Kühn and Osthus [17].

In this paper we consider multipartite packing, which restricts  $G$  to be a  $k$ -partite graph for  $k \geq 2$ . A  $k$ -partite graph is called *balanced* if its partition sets have the same size. Given a  $k$ -partite graph  $G$ , it is natural to consider the minimum partite degree  $\delta^*(G)$ , the minimum degree from a vertex in one partition set to any other partition set. When  $k = 2$ ,  $\delta^*(G)$  is simply  $\delta(G)$ . In most of the rest of this paper, the minimum degree condition stands for the minimum partite degree for short.

Let  $\mathcal{G}_k(n)$  denote the family of balanced  $k$ -partite graphs with  $n$  vertices in each of its partition sets. It is easy to see (e.g. using the König-Hall Theorem) that every bipartite graph  $G \in \mathcal{G}_2(n)$  with  $\delta^*(G) \geq n/2$  contains a 1-factor. Fischer [5] conjectured that if  $G \in \mathcal{G}_k(n)$  satisfies

$$\delta^*(G) \geq \frac{k-1}{k}n, \quad (1)$$

then  $G$  contains a  $K_k$ -factor and proved the existence of an *almost*  $K_k$ -factor for  $k = 3, 4$ . Magyar and Martin [20] noticed that the condition (1) is not sufficient for odd  $k$  and instead proved the following theorem for  $k = 3$ . (They actually showed that when  $n$  is divisible by 3, there is only one graph in  $\mathcal{G}_3(n)$ , denoted by  $\Gamma_3(n/3)$ , that satisfies (1) but fails to contain a  $K_3$ -factor, and adding any new edge to  $\Gamma_3(n/3)$  results in a  $K_3$ -factor.)

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**Theorem 1** ([20]). *There exists an integer  $n_0$  such that If  $n \geq n_0$  and  $G \in \mathcal{G}_3(n)$  satisfies  $\delta^*(G) \geq 2n/3 + 1$ , then  $G$  contains a  $K_3$ -factor.*

On the other hand, Martin and Szemerédi [21] proved the original conjecture holds for  $k = 4$ .

**Theorem 2** ([21]). *There exists an integer  $n_0$  such that if  $n \geq n_0$  and  $G \in \mathcal{G}_4(n)$  satisfies  $\delta^*(G) \geq 3n/4$ , then  $G$  contains a  $K_4$ -factor.*

Recently Keevash and Mycroft [9] and independently Lo and Markström [19] proved that Fischer's conjecture is asymptotically true, namely,  $\delta^*(G) \geq \frac{k-1}{k}n + o(n)$  guarantees a  $K_k$ -factor for all  $k \geq 3$ . Very recently, Keevash and Mycroft [10] improved this to an exact result.

In this paper we give a new proof of Theorems 1 and 2 by the absorbing method. Our approach is similar to that of [19] (in contrast, a geometric approach was employed in [9]). However, in order to prove exact results by the absorbing lemma, one needs only assume  $\delta^*(G) \geq (1 - 1/k)n$ , instead of  $\delta^*(G) \geq (1 - 1/k + \alpha)n$  for some  $\alpha > 0$  as in [19]. In fact, our absorbing lemma uses an even weaker assumption  $\delta^*(G) \geq (1 - 1/k - \alpha)n$  and has a more complicated absorbing structure.

The absorbing method, initiated by Rödl, Ruciński, and Szemerédi [23], has been shown to be effective handling extremal problems in graphs and hypergraphs. One example is the re-proof of Posa's conjecture by Levitt, Sárközy, and Szemerédi [18], while the original proof of Komlós, Sárközy, and Szemerédi [13] used the Regularity Lemma. Our paper is another example of replacing the regularity method with the absorbing method. Compared with the threshold  $n_0$  in Theorems 1 and 2 derived from the Regularity Lemma, the value of our  $n_0$  is much smaller.

Before presenting our proof, let us first recall the approach used in [20, 21]. Given a  $k$ -partite graph  $G \in \mathcal{G}_k(n)$  with parts  $V_1, \dots, V_k$ , the authors said that  $G$  is  $\Delta$ -extremal if each  $V_i$  contains a subset  $A_i$  of size  $\lfloor n/k \rfloor$  such that the density  $d(A_i, A_j) \leq \Delta$  for all  $i \neq j$ . Using standard but involved graph theoretic arguments, they solved the extremal case for  $k = 3, 4$  [20, Theorem 3.1], [21, Theorem 2.1].

**Theorem 3.** *Let  $k = 3, 4$ . There exists  $\Delta$  and  $n_0$  such that the following holds. Let  $n \geq n_0$  and  $G \in \mathcal{G}_k(n)$  be a  $k$ -partite graph satisfying  $\delta^*(G) \geq (2/3)n + 1$  when  $k = 3$  and (1) when  $k = 4$ . If  $G$  is  $\Delta$ -extremal, then  $G$  contains a  $K_k$ -factor.*

To handle the non-extremal case, they proved the following lemma ([20, Lemma 2.2] and [21, Lemma 2.2]).

**Lemma 4** (Almost Covering Lemma). *Let  $k = 3, 4$ . Given  $\Delta > 0$ , there exists  $\alpha > 0$  such that for every graph  $G \in \mathcal{G}_k(n)$  with  $\delta^*(G) \geq (1 - 1/k)n - \alpha n$  either  $G$  contains an almost  $K_k$ -factor that leaves at most  $C = C(k)$  vertices uncovered or  $G$  is  $\Delta$ -extremal.*

To improve the almost  $K_k$ -factor obtained from Lemma 4, they used the Regularity Lemma and Blow-up Lemma [14]. Here is where we need our absorbing lemma whose proof is given in Section 2. Our lemma actually gives a more detailed structure than what is needed for the extremal case when  $G$  does not satisfy the absorbing property.

We need some definitions. Given positive integers  $k$  and  $r$ , let  $\Theta_{k \times r}$  denote the graph with vertices  $a_{ij}$ ,  $i = 1, \dots, k$ ,  $j = 1, \dots, r$ , and  $a_{ij}$  is adjacent to  $a_{i'j'}$  if and only if  $i \neq i'$  and  $j \neq j'$ . In addition, given a positive integer  $t$ , the graph  $\Theta_{k \times r}(t)$  denotes the blow-up of  $\Theta_{k \times r}$ , obtained by replacing vertices  $a_{ij}$  with sets  $A_{ij}$  of size  $t$ , and edges  $a_{ij}a_{i'j'}$  with complete bipartite graphs between  $A_{ij}$  and  $A_{i'j'}$ . Given  $\epsilon, \Delta > 0$  and  $t \geq 1$  (not necessarily an integer), we say that a  $k$ -partite graph  $G$  is  $(\epsilon, \Delta)$ -approximate to  $\Theta_{k \times r}(t)$  if each of its partition sets  $V_i$  can be partitioned into  $\bigcup_{j=1}^r V_{ij}$  such that  $||V_{ij}| - t| \leq \epsilon t$  for all  $i, j$  and  $d(V_{ij}, V_{i'j}) \leq \Delta$  whenever  $i \neq i'$ .<sup>1</sup>

<sup>1</sup>Here we follow the definition of  $(\epsilon, \Delta)$ -approximation in [20, 21]. It seems natural to require that  $d(V_{ij}, V_{i'j'}) \geq 1 - \Delta$  whenever  $i \neq i'$  and  $j \neq j'$  as well. However, this follows from  $d(V_{ij}, V_{i'j}) \leq \Delta$  ( $i \neq i'$ ) when  $\delta^*(G) \geq (1 - 1/r)rt$ .

**Lemma 5** (Absorbing Lemma). *Given  $k \geq 3$  and  $\Delta > 0$ , there exists  $\alpha = \alpha(k, \Delta) > 0$  and an integer  $n_1 > 0$  such that the following holds. Let  $n \geq n_1$  and  $G \in \mathcal{G}_k(n)$  be a  $k$ -partite graph on  $V_1 \cup \dots \cup V_k$  such that  $\delta^*(G) \geq (1 - 1/k)n - \alpha n$ . Then one of the following cases holds.*

- (1)  *$G$  contains a  $K_k$ -matching  $M$  of size  $|M| \leq 2(k-1)\alpha^{4k-2}n$  in  $G$  such that for every  $W \subset V \setminus V(M)$  with  $|W \cap V_1| = \dots = |W \cap V_k| \leq \alpha^{8k-6}n/4$ , there exists a  $K_k$ -matching covering exactly the vertices in  $V(M) \cup W$ .*
- (2) *We may remove some edges from  $G$  so that the resulting graph  $G'$  satisfies  $\delta^*(G') \geq (1 - 1/k)n - \alpha n$  and is  $(\Delta/6, \Delta/2)$ -approximate to  $\Theta_{k \times k}(\frac{n}{k})$ .*

The  $K_k$ -matching  $M$  in Lemma 5 has the so-called *absorbing* property: it can absorb *any* balanced set with a much smaller size.

*Proof of Theorems 1 and 2.* Let  $k = 3, 4$ . Let  $\alpha \ll \Delta$ , where  $\Delta$  is given by Theorem 3 and  $\alpha$  satisfies both Lemmas 4 and 5. Suppose that  $n$  is sufficiently large. Let  $G \in \mathcal{G}_k(n)$  be a  $k$ -partite graph satisfying  $\delta^*(G) \geq (2/3)n + 1$  when  $k = 3$  and (1) when  $k = 4$ . By Lemma 5, either  $G$  contains a subgraph which is  $(\Delta/6, \Delta/2)$ -approximate to  $\Theta_{k \times k}(\frac{n}{k})$  or  $G$  contains an absorbing  $K_k$ -matching  $M$ . In the former case, for  $i = 1, \dots, k$ , we add or remove at most  $\frac{\Delta n}{6k}$  vertices from  $V_{i1}$  to obtain a set  $A_i \subset V_i$  of size  $\lfloor n/k \rfloor$ . For  $i \neq i'$ , we have

$$\begin{aligned} e(A_i, A_{i'}) &\leq e(V_{i1}, V_{i'1}) + \frac{\Delta n}{6k}(|A_i| + |A_{i'}|) \\ &\leq \frac{\Delta}{2}|V_{i1}||V_{i'1}| + 2\frac{\Delta n}{6k}\left\lfloor \frac{n}{k} \right\rfloor \\ &\leq \frac{\Delta}{2}\left(1 + \frac{\Delta}{6}\right)^2\left(\frac{n}{k}\right)^2 + \frac{\Delta n}{3k}\left\lfloor \frac{n}{k} \right\rfloor \\ &\leq \Delta\left\lfloor \frac{n}{k} \right\rfloor\left\lfloor \frac{n}{k} \right\rfloor, \end{aligned}$$

which implies that  $d(A_i, A_{i'}) \leq \Delta$ . Thus  $G$  is  $\Delta$ -extremal. By Theorem 3,  $G$  contains a  $K_k$ -factor. In the latter case,  $G$  contains a  $K_k$ -matching  $M$  of size  $|M| \leq 2(k-1)\alpha^{4k-2}n$  such that for every  $W \subset V \setminus V(M)$  with  $|W \cap V_1| = \dots = |W \cap V_k| \leq \alpha^{8k-6}n/4$ , there exists a  $K_k$ -matching on  $V(M) \cup W$ . Let  $G' = G \setminus V(M)$  be the induced subgraph of  $G$  on  $V(G) \setminus V(M)$ , and  $n' = |V(G')|$ . Clearly  $G'$  is balanced. As  $\alpha \ll 1$ , we have

$$\delta^*(G') \geq \delta^*(G) - |M| \geq \left(1 - \frac{1}{k}\right)n - 2(k-1)\alpha^{4k-2}n \geq \left(1 - \frac{1}{k} - \alpha\right)n'.$$

By Lemma 4,  $G'$  contains a  $K_k$ -matching  $M'$  such that  $|V(G') \setminus V(M')| \leq C$ . Let  $W = V(G') \setminus V(M')$ . Clearly  $|W \cap V_1| = \dots = |W \cap V_k| \leq C/k$ . Since  $C/k \leq \alpha^{8k-6}n/4$  for sufficiently large  $n$ , by the absorbing property of  $M$ , there is a  $K_k$ -matching  $M''$  on  $V(M) \cup W$ . This gives the desired  $K_k$ -factor  $M' \cup M''$  of  $G$ .  $\square$

#### Remarks.

- Since our Lemma 5 works for all  $k \geq 3$ , it has the potential of proving a general multipartite Hajnal-Szemerédi theorem. To do it, one only needs to prove Theorem 3 and Lemma 4 for  $k \geq 5$ .
- Since our Lemma 5 gives a detailed structure of  $G$  when  $G$  does not have desired absorbing  $K_k$ -matching, it has the potential of simplifying the proof of the extremal case. Indeed, if one can refine Lemma 4 such that it concludes that  $G$  either contains an almost  $K_k$ -factor or it is approximate to  $\Theta_{k \times k}(\frac{n}{k})$  and other extremal graphs, then in Theorem 3 we may assume that  $G$  is actually approximate to these extremal graphs.
- Using the Regularity Lemma, researchers have obtained results on packing arbitrary graphs in  $k$ -partite graphs, see [26, 8, 3, 2] for  $k = 2$  and [22] for  $k = 3$ . With the help of the recent

result of Keevash–Mycroft [9] and Lo-Markström [19], it seems not very difficult to extend these results to the  $k \geq 4$  case (though exact results may be much harder). However, it seems difficult to replace the regularity method by the absorbing method for these problems.

## 2. PROOF OF THE ABSORBING LEMMA

In this section we prove the Absorbing Lemma (Lemma 5). We first introduce the concepts of reachability.

**Definition 6.** In a graph  $G$ , a vertex  $x$  is reachable from another vertex  $y$  by a set  $S \subseteq V(G)$  if both  $G[x \cup S]$  and  $G[y \cup S]$  contain  $K_k$ -factors. In this case, we say  $S$  connects  $x$  and  $y$ .

The following lemma plays a key role in constructing absorbing structures. We postpone its proof to the end of the section.

**Lemma 7** (Reachability Lemma). *Given  $k \geq 3$  and  $\Delta > 0$ , there exists  $\alpha = \alpha(k, \Delta) > 0$  and an integer  $n_2 > 0$  such that the following holds. Let  $n \geq n_2$  and  $G \in \mathcal{G}_k(n)$  be a  $k$ -partite graph on  $V_1 \cup \dots \cup V_k$  such that  $\delta^*(G) \geq (1 - 1/k)n - \alpha n$ . Then one of the following cases holds.*

- (1) *For any  $x$  and  $y$  in  $V_i$ ,  $i \in [k]$ ,  $x$  is reachable from  $y$  by either at least  $\alpha^3 n^{k-1}$   $(k-1)$ -sets or at least  $\alpha^3 n^{2k-1}$   $(2k-1)$ -sets in  $G$ .*
- (2) *We may remove some edges from  $G$  so that the resulting graph  $G'$  satisfies  $\delta^*(G') \geq (1 - 1/k)n - \alpha n$  and is  $(\Delta/6, \Delta/2)$ -approximate to  $\Theta_{k \times k}(\frac{n}{k})$ .*

With the aid of Lemma 7, the proof of Lemma 5 becomes standard counting and probabilistic arguments, as shown in [7].

*Proof of Lemma 5.* We assume that  $G$  does not satisfy the second property stated in the lemma.

Given a crossing  $k$ -tuple  $T = (v_1, \dots, v_k)$ , with  $v_i \in V_i$ , for  $i = 1, \dots, k$ , we call a set  $A$  an *absorbing set* for  $T$  if both  $G[A]$  and  $G[A \cup T]$  contain  $K_k$ -factors. Let  $\mathcal{L}(T)$  denote the family of all  $2k(k-1)$ -sets that absorb  $T$  (the reason why our absorbing sets are of size  $2k(k-1)$  can be seen from the proof of Claim 8 below).

**Claim 8.** *For every crossing  $k$ -tuple  $T$ , we have  $|\mathcal{L}(T)| > \alpha^{4k-3} n^{2k(k-1)}$ .*

*Proof.* Fix a crossing  $k$ -tuple  $T$ . First we try to find a copy of  $K_k$  containing  $v_1$  and avoiding  $v_2, \dots, v_k$ . By the minimum degree condition, there are at least

$$\prod_{i=2}^k \left( n - 1 - (i-1) \left( \frac{1}{k} + \alpha \right) n \right) \geq \prod_{i=2}^k \left( n - (i-1) \frac{n}{k} - ((k-1)\alpha n + 1) \right)$$

such copies of  $K_k$ . When  $n \geq 3k^2$  and  $\frac{1}{\alpha} \geq 3k^2$ , we have  $(k-1)\alpha n + 1 \leq n/(3k)$  and thus the number above is at least

$$\prod_{i=2}^k \left( n - (i-1) \frac{n}{k} - \frac{n}{3k} \right) \geq \left( \frac{n}{k} \right)^{k-1}, \text{ when } k \geq 3.$$

Fix such a copy of  $K_k$  on  $\{v_1, u_2, u_3, \dots, u_k\}$ . Consider  $u_2$  and  $v_2$ . By Lemma 7 and the assumption that  $G$  does not satisfy the second property of the lemma, we can find at least  $\alpha^3 n^{k-1}$   $(k-1)$ -sets or  $\alpha^3 n^{2k-1}$   $(2k-1)$ -sets to connect  $u_2$  and  $v_2$ . If  $S$  is a  $(k-1)$ -set that connects  $u_2$  and  $v_2$ , then  $S \cup K$  also connects  $u_2$  and  $v_2$  for any  $k$ -set  $K$  such that  $G[K] \cong K_k$  and  $K \cap S = \emptyset$ . There are at least

$$(n-2) \prod_{i=2}^k \left( n - 1 - (i-1) \left( \frac{1}{k} + \alpha \right) n \right) \geq \frac{n}{2} \left( \frac{n}{k} \right)^{k-1}$$

copies of  $K_k$  in  $G$  avoiding  $u_2$ ,  $v_2$  and  $S$ . If there are at least  $\alpha^3 n^{k-1}$   $(k-1)$ -sets that connect  $u_2$  and  $v_2$ , then at least

$$\alpha^3 n^{k-1} \cdot \frac{n}{2} \left(\frac{n}{k}\right)^{k-1} \frac{1}{\binom{2k-1}{k-1}} \geq 2\alpha^4 n^{2k-1}$$

$(2k-1)$ -sets connect  $u_2$  and  $v_2$  because a  $(2k-1)$ -set can be counted at most  $\binom{2k-1}{k-1}$  times. Since  $2\alpha^4 < \alpha^3$ , we can assume that there are always at least  $2\alpha^4 n^{2k-1}$   $(2k-1)$ -sets connecting  $u_2$  and  $v_2$ . We inductively choose disjoint  $(2k-1)$ -sets that connects  $v_i$  and  $u_i$  for  $i = 2, \dots, k$ . For each  $i$ , we must avoid  $T$ ,  $u_2, \dots, u_k$ , and  $i-2$  previously selected  $(2k-1)$ -sets. Hence there are at least  $2\alpha^4 n^{2k-1} - (2k-1)(i-1)n^{2k-2} > \alpha^4 n^{2k-1}$  choices of such  $(2k-1)$ -sets for each  $i \geq 2$ . Putting all these together, and using the assumption that  $\alpha$  is sufficiently small, we have

$$|\mathcal{L}(T)| \geq \left(\frac{n}{k}\right)^{k-1} \cdot (\alpha^4 n^{2k-1})^{k-1} > \alpha^{4k-3} n^{2k(k-1)}.$$

□

Every set  $S \in \mathcal{L}(T)$  is *balanced* because  $G[S]$  contains a  $K_k$ -factor and thus  $|S \cap V_1| = \dots = |S \cap V_k| = 2(k-1)$ . Note that there are  $\binom{n}{2(k-1)}^k$  balanced  $2k(k-1)$ -sets in  $G$ . Let  $\mathcal{F}$  be the random family of  $2k(k-1)$ -sets obtained by selecting each balanced  $2k(k-1)$ -set from  $V(G)$  independently with probability  $p := \alpha^{4k-3} n^{1-2k(k-1)}$ . Then by Chernoff's bound, since  $n$  is sufficiently large, with probability  $1 - o(1)$ , the family  $\mathcal{F}$  satisfies the following properties:

$$|\mathcal{F}| \leq 2\mathbb{E}(|\mathcal{F}|) \leq 2p \binom{n}{2(k-1)}^k \leq \alpha^{4k-2} n, \quad (2)$$

$$|\mathcal{L}(T) \cap \mathcal{F}| \geq \frac{1}{2}\mathbb{E}(|\mathcal{L}(T) \cap \mathcal{F}|) \geq \frac{1}{2}p|\mathcal{L}(T)| \geq \frac{\alpha^{8k-6}n}{2} \text{ for every crossing } k\text{-tuple } T. \quad (3)$$

Let  $Y$  be the number of intersecting pairs of members of  $\mathcal{F}$ . Since each fixed balanced  $2k(k-1)$ -set intersects at most  $2k(k-1)\binom{n-1}{2(k-1)-1}\binom{n}{2(k-1)}^{k-1}$  other balanced  $2k(k-1)$ -sets in  $G$ ,

$$\mathbb{E}(Y) \leq p^2 \binom{n}{2(k-1)}^k 2k(k-1) \binom{n-1}{2k-3} \binom{n}{2(k-1)}^{k-1} \leq \frac{1}{8} \alpha^{8k-6} n.$$

By Markov's bound, with probability at least  $\frac{1}{2}$ ,  $Y \leq \alpha^{8k-6} n/4$ . Therefore, we can find a family  $\mathcal{F}$  satisfying (2), (3) and having at most  $\alpha^{8k-6} n/4$  intersecting pairs. Remove one set from each of the intersecting pairs and the sets that have no  $K_k$ -factor from  $\mathcal{F}$ , we get a subfamily  $\mathcal{F}'$  consisting of pairwise disjoint absorbing  $2k(k-1)$ -sets which satisfies  $|\mathcal{F}'| \leq |\mathcal{F}| \leq \alpha^{4k-2} n$  and for all crossing  $T$ ,

$$|\mathcal{L}(T) \cap \mathcal{F}'| \geq \frac{\alpha^{8k-6}n}{2} - \frac{\alpha^{8k-6}n}{4} \geq \frac{\alpha^{8k-6}n}{4}.$$

Since  $\mathcal{F}'$  consists of disjoint absorbing sets and each absorbing set is covered by a  $K_k$ -matching,  $V(\mathcal{F}')$  is covered by some  $K_k$ -matching  $M$ . Since  $|\mathcal{F}'| \leq \alpha^{4k-2} n$ , we have  $|M| \leq 2k(k-1)\alpha^{4k-2} n/k = 2(k-1)\alpha^{4k-2} n$ . Now consider a balanced set  $W \subseteq V(G) \setminus V(\mathcal{F}')$  such that  $|W \cap V_1| = \dots = |W \cap V_k| \leq \alpha^{8k-6} n/4$ . Arbitrarily partition  $W$  into at most  $\alpha^{8k-6} n/4$  crossing  $k$ -tuples. We absorb each of the  $k$ -tuples with a different  $2k(k-1)$ -set from  $\mathcal{L}(T) \cap \mathcal{F}'$ . As a result,  $V(\mathcal{F}') \cup W$  is covered by a  $K_k$ -matching, as desired.

□

The rest of the paper is devoted to proving Lemma 7. First we prove a useful lemma. A weaker version of it appears in [21, Proposition 1.4] with a brief proof sketch.

**Lemma 9.** *Let  $k \geq 2$  be an integer,  $t \geq 1$  and  $\epsilon \ll 1$ . Let  $H$  be a  $k$ -partite graph on  $V_1 \cup \dots \cup V_k$  such that  $|V_i| \geq (k-1)(1-\epsilon)t$  for all  $i$  and each vertex is nonadjacent to at most  $(1+\epsilon)t$  vertices*

in each of the other color classes. Then either  $H$  contains at least  $\epsilon^2 t^k$  copies of  $K_k$ , or  $H$  is  $(16k^4\epsilon^{1/2^{k-2}}, 16k^4\epsilon^{1/2^{k-2}})$ -approximate to  $\Theta_{k \times (k-1)}(t)$ .

*Proof.* First we derive an upper bound for  $|V_i|$ ,  $i \in [k]$ . Suppose for example, that  $|V_k| \geq (k-1)(1+\epsilon)t + \epsilon t$ . Then if we greedily construct copies of  $K_k$  while choosing the last vertex from  $V_k$ , by the minimum degree condition and  $\epsilon \ll 1$ , there are at least

$$\begin{aligned} & |V_1| \cdot (|V_2| - (1+\epsilon)t) \cdots (|V_{k-1}| - (k-2)(1+\epsilon)t) \cdot (|V_k| - (k-1)(1+\epsilon)t) \\ & \geq (k-1)(1-\epsilon)t \cdot (k-2-k\epsilon)t \cdots (1-(2k-3)\epsilon)t \cdot \epsilon t \\ & \geq (k-1-\frac{1}{2})(k-2-\frac{1}{2}) \cdots (1-\frac{1}{2})\epsilon t^k \geq \frac{\epsilon}{2}t^k \end{aligned}$$

copies of  $K_k$  in  $H$ , so we are done. We thus assume that for all  $i$ ,

$$|V_i| \leq (k-1)(1+\epsilon)t + \epsilon t < (k-1)(1+2\epsilon)t. \quad (4)$$

Now we proceed by induction on  $k$ . The base case is  $k=2$ . If  $H$  has at least  $\epsilon^2 t^2$  edges, then we are done. Otherwise  $e(H) < \epsilon^2 t^2$ . Using the lower bound for  $|V_i|$ , we obtain that

$$d(V_1, V_2) < \frac{\epsilon^2 t^2}{|V_1| \cdot |V_2|} \leq \frac{\epsilon^2}{(1-\epsilon)^2} < \epsilon.$$

Hence  $H$  is  $(2\epsilon, \epsilon)$ -approximate to  $\Theta_{2 \times 1}(t)$ . When  $k=2$ ,  $16k^4\epsilon^{1/2^{k-2}} = 256\epsilon$ , so we are done.

Now assume that  $k \geq 3$  and the conclusion holds for  $k-1$ . Let  $H$  be a  $k$ -partite graph satisfying the assumptions and assume that  $H$  contains less than  $\epsilon^2 t^k$  copies of  $K_k$ .

For simplicity, write  $N_i(v) = N(v) \cap V_i$  for any vertex  $v$ . Let  $V'_1 \subset V_1$  be the vertices which are in at least  $\epsilon t^{k-1}$  copies of  $K_k$  in  $H$ , and let  $\tilde{V}_1 = V_1 \setminus V'_1$ . Note that  $|V'_1| < \epsilon t$  otherwise we get at least  $\epsilon^2 t^k$  copies of  $K_k$  in  $H$ . Fix  $v_0 \in \tilde{V}_1$ . For  $2 \leq i \leq k$ , by the minimum degree condition and  $k \geq 3$ ,

$$|N_i(v_0)| \geq (k-1)(1-\epsilon)t - (1+\epsilon)t = (k-2) \left(1 - \frac{k}{k-2}\epsilon\right)t \geq (k-2)(1-3\epsilon)t.$$

On the other hand, following the same arguments as we used for (4), we derive that

$$|N_i(v_0)| \leq (k-2)(1+2\epsilon t). \quad (5)$$

The minimum degree condition implies that a vertex in  $N(v_0)$  misses at most  $(1+\epsilon)t$  vertices in each  $N_i(v_0)$ . We now apply induction with  $k-1$ ,  $t$  and  $3\epsilon$  on  $H[N(v_0)]$ . Because of the definition of  $V'_1$ , we conclude that  $N(v_0)$  is  $(\epsilon', \epsilon')$ -approximate to  $\Theta_{(k-1) \times (k-2)}(t)$ , where

$$\epsilon' := 16(k-1)^4(3\epsilon)^{1/2^{k-3}}.$$

This means that we can partition  $N_i(v_0)$  into  $A_{i1} \cup \cdots \cup A_{i(k-2)}$  for  $2 \leq i \leq k$  such that

$$\forall 2 \leq i \leq k, 1 \leq j \leq k-2, (1-\epsilon')t \leq |A_{ij}| \leq (1+\epsilon')t \quad \text{and} \quad (6)$$

$$\forall 2 \leq i < i' \leq k, 1 \leq j \leq k-2, d(A_{ij}, A_{i'j}) \leq \epsilon'. \quad (7)$$

Furthermore, let  $A_{i(k-1)} := V_i \setminus N(v_0)$  for  $i = 2, \dots, k$ . By (5) and the minimum degree condition, we get that

$$(1-(3k-5)\epsilon)t \leq |A_{i(k-1)}| \leq (1+\epsilon)t, \quad (8)$$

for  $i = 2, \dots, k$ .

Let  $A_{ij}^c = V_i \setminus A_{ij}$  denote the complement of  $A_{ij}$ . Let  $\bar{e}(A, B) = |A||B| - e(A, B)$  denote the number of non-edges between two disjoint sets  $A$  and  $B$ , and  $\bar{d}(A, B) = \bar{e}(A, B)/(|A||B|) = 1 - d(A, B)$ . Given two disjoint sets  $A$  and  $B$  (with density close to one) and  $\alpha > 0$ , we call a vertex  $a \in A$  is  $\alpha$ -typical to  $B$  if  $\deg_B(a) \geq (1-\alpha)|B|$ .

**Claim 10.** Let  $2 \leq i \neq i' \leq k$ ,  $1 \leq j \neq j' \leq k-1$ .

$$(1) \quad d(A_{ij}, A_{i'j'}) \geq 1-3\epsilon' \quad \text{and} \quad d(A_{ij}, A_{i'j}^c) \geq 1-3\epsilon'.$$

- (2) All but at most  $\sqrt{3\epsilon'}$  vertices in  $A_{ij}$  are  $\sqrt{3\epsilon'}$ -typical to  $A_{i'j'}$ ; at most  $\sqrt{3\epsilon'}$  vertices in  $A_{ij}$  are  $\sqrt{3\epsilon'}$ -typical to  $A_{i'j}$ .

*Proof.* (1). Since  $A_{i'j}^c = \bigcup_{j' \neq j} A_{i'j'}$ , the second assertion  $d(A_{ij}, A_{i'j}^c) \geq 1 - 3\epsilon'$  immediately follows from the first assertion  $d(A_{ij}, A_{i'j'}) \geq 1 - 3\epsilon'$ . Thus it suffices to show that  $d(A_{ij}, A_{i'j'}) \geq 1 - 3\epsilon'$ , or equivalently that  $\bar{d}(A_{ij}, A_{i'j'}) \leq 3\epsilon'$ .

Assume  $j \geq 2$ . By (7), we have  $e(A_{ij}, A_{i'j}) \leq \epsilon'|A_{ij}||A_{i'j}|$ . So  $\bar{e}(A_{ij}, A_{i'j}) \geq (1 - \epsilon')|A_{ij}||A_{i'j}|$ . By the minimum degree condition and (6),

$$\begin{aligned} \bar{e}(A_{ij}, A_{i'j}^c) &\leq [(1 + \epsilon)t - (1 - \epsilon')|A_{i'j}|]|A_{ij}| \\ &\leq [(1 + \epsilon)t - (1 - \epsilon')(1 - \epsilon')t]|A_{ij}| \\ &< (\epsilon + 2\epsilon')t|A_{ij}|, \end{aligned}$$

which implies that  $\bar{e}(A_{ij}, A_{i'j'}) \leq (\epsilon + 2\epsilon')t|A_{ij}|$  for any  $j' \neq j$  and  $1 \leq j' \leq k - 1$ . By (6) and (8), we have  $|A_{i'j'}| \geq (1 - \epsilon')t$ . Hence

$$\bar{d}(A_{ij}, A_{i'j'}) \leq (\epsilon + 2\epsilon') \frac{t}{|A_{i'j'}|} \leq (\epsilon + 2\epsilon') \frac{t}{(1 - \epsilon')t} \leq 3\epsilon',$$

where the last inequality holds because  $\epsilon \ll \epsilon' \ll 1$ .

(2) Given two disjoint sets  $A$  and  $B$ , if  $\bar{d}(A, B) \leq \alpha$  for some  $\alpha > 0$ , then at most  $\sqrt{\alpha}|A|$  vertices  $a \in A$  satisfy  $\deg_B(a) < (1 - \sqrt{\alpha})|B|$ . Hence Part (2) immediately follows from Part (1).  $\square$

We need a lower bound for the number of copies of  $K_k$  in a dense  $k$ -partite graph.

**Proposition 11.** *Let  $G$  be a  $k$ -partite graph with vertex class  $V_1, \dots, V_k$ . Suppose for every two vertex classes, the pairwise density  $d(V_i, V_j) \geq 1 - \alpha$  for some  $\alpha \leq (k + 1)^{-4}$ , then there are at least  $\frac{1}{2} \prod_i |V_i|$  copies of  $K_k$  in  $G$ .*

*Proof.* Given two disjoint sets  $V_i$  and  $V_j$ , if  $\bar{d}(V_i, V_j) \leq \alpha$  for some  $\alpha > 0$ , then at most  $\sqrt{\alpha}|V_i|$  vertices  $v \in V_i$  satisfy  $\deg_{V_j}(v) < (1 - \sqrt{\alpha})|V_j|$ . Thus, by choosing typical vertices greedily and the assumption  $\alpha \leq (k + 1)^{-4}$ , there are at least

$$(1 - \sqrt{\alpha})|V_1|(1 - 2\sqrt{\alpha})|V_2| \cdots (1 - k\sqrt{\alpha})|V_k| > (1 - (1 + \cdots + k)\sqrt{\alpha}) \prod_i |V_i| > \frac{1}{2} \prod_i |V_i|$$

copies of  $K_k$  in  $G$ .  $\square$

Let  $\epsilon'' = 2k\sqrt{\epsilon'}$ . Now we want to study the structure of  $\tilde{V}_1$ .

**Claim 12.** *Given  $v \in \tilde{V}_1$  and  $2 \leq i \leq k$ , there exists  $j \in [k - 1]$ , such that  $|N_{A_{ij}}(v)| < \epsilon''t$ .*

*Proof.* Suppose instead, that there exist  $v \in \tilde{V}_1$  and some  $2 \leq i_0 \leq k$ , such that  $|N_{A_{i_0j}}(v)| \geq \epsilon''t$  for all  $j \in [k - 1]$ . By the minimum degree condition, for each  $2 \leq i \leq k$ , there is at most one  $j \in [k - 1]$  such that  $|N_{A_{ij}}(v)| < t/3$ . Therefore we can greedily choose  $k - 2$  distinct  $j_i$  for  $i \neq i_0$ , such that  $|N_{A_{ij_i}}(v)| \geq t/3$ . Let  $j_{i_0}$  be the (unique) unused index. Note that

$$\forall i \neq i_0, \quad \frac{|A_{ij_i}|}{|N_{A_{ij_i}}(v)|} \leq \frac{(1 + \epsilon')t}{t/3} < 4, \quad \text{and} \quad \frac{|A_{i_0j_{i_0}}|}{|N_{A_{i_0j_{i_0}}}(v)|} \leq \frac{(1 + \epsilon')t}{\epsilon''t} < \frac{2}{\epsilon''}$$

So for any  $i \neq i'$ , by Claim 10 and the definition of  $\epsilon''$ , we have

$$\bar{d}(N_{A_{ij_i}}(v), N_{A_{i'j_{i'}}}(v)) \leq \frac{3\epsilon'|A_{ij_i}||A_{i'j_{i'}}|}{|N_{A_{ij_i}}(v)||N_{A_{i'j_{i'}}}(v)|} \leq 3\epsilon' \cdot 4 \cdot \frac{2}{\epsilon''} = \frac{6}{k^2}\epsilon''. \quad (9)$$

Since  $\epsilon \ll \epsilon'' \ll 1$ , by Proposition 11, there are at least

$$\frac{1}{2} \prod_i N_{A_{ij_i}}(v) \geq \frac{1}{2} \cdot \epsilon'' t \left(\frac{t}{3}\right)^{k-2} = \frac{\epsilon''}{2 \cdot 3^{k-2}} t^{k-1} > \epsilon t^{k-1}$$

copies of  $K_{k-1}$  in  $N(v)$ , contradicting the assumption  $v \in \tilde{V}_1$ .  $\square$

Note that if  $\deg_{A_{ij}}(v) < \epsilon'' t$ , at least  $|A_{ij}| - \epsilon'' t$  vertices of  $A_{ij}$  are not in  $N(v)$ . By the minimum degree condition, (6) and (8), it follows that

$$|A_{ij}^c \setminus N(v)| \leq (1 + \epsilon)t - (|A_{ij}| - \epsilon'' t) \leq (1 + \epsilon)t - (1 - \epsilon')t + \epsilon'' t \leq 2\epsilon'' t. \quad (10)$$

Fix a vertex  $v \in \tilde{V}_1$ . Given  $2 \leq i \leq k$ , let  $\ell_i$  denote the (unique) index such that  $|N_{A_{i\ell_i}}(v)| < \epsilon'' t$  (the existence of  $\ell_i$  follows from Claim 12).

**Claim 13.** *We have  $\ell_2 = \ell_3 = \dots = \ell_k$ .*

*Proof.* Otherwise, say  $\ell_2 \neq \ell_3$ , then we set  $j_2 = \ell_3$  and for  $3 \leq i \leq k$ , greedily choose distinct  $j_k, j_{k-1}, \dots, j_3 \in [k-1] \setminus \{\ell_3\}$  such that  $j_i \neq \ell_i$  (this is possible as  $j_3$  is chosen at last). Let us bound the number of copies of  $K_{k-1}$  in  $\bigcup_{i=2}^k N_{A_{ij_i}}(v)$ . By 10, we get  $|N_{A_{ij_i}}(v)| \geq |A_{ij_i}| - 2\epsilon'' t \geq t/2$  for all  $i$ . As in (9), for any  $i \neq i'$ , we derive that  $\bar{d}(N_{A_{ij_i}}(v), N_{A_{i'j_{i'}}}(v)) \leq 3\epsilon'' \cdot 4 \cdot 4 = 48\epsilon''$ . Since  $\epsilon'' \ll 1$ , by Proposition 11, we get at least  $\frac{1}{2} \left(\frac{t}{2}\right)^{k-1} > \epsilon t^{k-1}$  copies of  $K_{k-1}$  in  $N(v)$ , a contradiction.  $\square$

We define  $A_{1j} := \{v \in \tilde{V}_1 : |N_{A_{2j}}(v)| < \epsilon'' t\}$  for  $j \in [k-1]$ . By Claims 12 and 13, this yields a partition of  $\tilde{V}_1 = \bigcup_{j=1}^{k-1} A_{1j}$  such that

$$d(A_{1j}, A_{ij}) < \frac{\epsilon'' t |A_{1j}|}{|A_{1j}| |A_{ij}|} \leq \frac{\epsilon'' t}{(1 - \epsilon')t} < (1 + 2\epsilon')\epsilon'' \quad \text{for } i \geq 2 \text{ and } j \geq 1. \quad (11)$$

By (6), (8) and (10), as  $(3k-5)\epsilon \leq \epsilon'$ , we have

$$\bar{d}(A_{1j}, A_{ij'}) < \frac{|A_{1j}| 2\epsilon'' t}{|A_{1j}| |A_{ij'}|} \leq \frac{2\epsilon'' t}{(1 - \epsilon')t} < 3\epsilon'' \quad \text{for } i \geq 2 \text{ and } j \neq j'. \quad (12)$$

We claim  $|A_{1j}| \leq (1 + \epsilon)t + (1 + 2\epsilon')\epsilon'' |A_{1j}|$  for all  $j$ . Otherwise, by the minimum degree condition, we have  $\deg_{A_{1j}}(v) > (1 + 2\epsilon')\epsilon'' |A_{1j}|$  for all  $v \in A_{1j}$ , and consequently  $d(A_{1j}, A_{ij}) > (1 + 2\epsilon')\epsilon''$ , contradicting (11). We thus conclude that

$$|A_{1j}| \leq \frac{1 + \epsilon}{1 - (1 + 2\epsilon')\epsilon''} t < (1 + 2\epsilon'')t. \quad (13)$$

Since  $|V'_1| \leq \epsilon t$ , we have  $|\bigcup_{j=1}^{k-1} A_{1j}| = |V_1 \setminus V'_1| \geq |V_1| - \epsilon t$ . Using (13), we now obtain a lower bound for  $|A_{1j}|$ ,  $j \in [k-1]$ :

$$|A_{1j}| \geq (k-1)(1 - \epsilon)t - (k-2)(1 + 2\epsilon'')t - \epsilon t \geq (1 - 2k\epsilon'')t. \quad (14)$$

It remains to show that for  $2 \leq i \neq i' \leq k$ ,  $d(A_{i(k-1)}, A_{i'(k-1)})$  is small. Write  $N(v_1 \dots v_m) = \bigcap_{1 \leq i \leq m} N(v_i)$ .

**Claim 14.**  *$d(A_{i(k-1)}, A_{i'(k-1)}) \leq 6\epsilon''$  for  $2 \leq i, i' \leq k$ .*

*Proof.* Suppose to the contrary, that say  $d(A_{(k-1)(k-1)}, A_{k(k-1)}) > 6\epsilon''$ . We first select  $k-2$  sets  $A_{ij}$  with  $1 \leq i \leq k-2$  and  $1 \leq j \leq k-2$  such that no two of them are on the same row or column – there are  $(k-2)!$  choices. Fix one of them, say  $A_{11}, A_{22}, \dots, A_{(k-2)(k-2)}$ . We construct copies of  $K_{k-2}$  in  $A_{11} \cup A_{22} \cup \dots \cup A_{(k-2)(k-2)}$  as follows. Pick arbitrary  $v_1 \in A_{11}$ . For  $2 \leq i \leq k-2$ , we select  $v_i \in N_{A_{ii}}(v_1 \dots v_{i-1})$  such that  $v_i$  is  $\sqrt{3\epsilon'}$ -typical to  $A_{(k-1)(k-1)}$ ,  $A_{k(k-1)}$  and all  $A_{jj}$ ,  $i < j \leq k-2$ . By Claim 10 and (10), there are at least  $(1 - (k-2)\sqrt{3\epsilon'})|A_{ii}| - 2\epsilon'' t \geq t/2$  choices for each  $v_i$ . After selecting  $v_1, \dots, v_{k-2}$ , we select adjacent vertices  $v_{k-1} \in A_{(k-1)(k-1)}$  and  $v_k \in A_{k(k-1)}$  such



that  $v_{k-1}, v_k \in N(v_1 \cdots v_{k-2})$ . For  $j \in \{k-1, k\}$ , we know that  $N(v_1)$  misses at most  $2\epsilon''t$  vertices in  $A_{j(k-1)}$ , and at most  $(k-3)\sqrt{3\epsilon'}|A_{j(k-1)}|$  vertices of  $A_{j(k-1)}$  are not in  $N(v_2 \cdots v_{k-2})$ . Since  $d(A_{(k-1)1}, A_{k1}) > 6\epsilon''$  and  $\epsilon'' = 2k\sqrt{\epsilon'}$ , there are at least

$$\begin{aligned} & 6\epsilon''|A_{(k-1)(k-1)}||A_{k(k-1)}| - 2\epsilon''t(|A_{(k-1)(k-1)}| + |A_{k(k-1)}|) - 2(k-3)\sqrt{3\epsilon'}|A_{(k-1)(k-1)}||A_{k(k-1)}| \\ & \geq (6\epsilon'' - 4\epsilon'' - 4(k-3)\sqrt{\epsilon'})|A_{(k-1)(k-1)}||A_{k(k-1)}| \\ & = 12\sqrt{\epsilon'}|A_{(k-1)(k-1)}||A_{k(k-1)}| \geq 6\sqrt{\epsilon'}t^2 \end{aligned}$$

such pairs  $v_{k-1}, v_k$ . Together with the choices of  $v_1, \dots, v_{k-2}$ , we obtain at least  $(k-2)!(\frac{t}{2})^{k-2} 6\sqrt{\epsilon'}t^2 > \epsilon t^k$  copies of  $K_k$ , a contradiction.  $\square$

In summary, by (6), (8), (13) and (14), we have  $(1 - 2k\epsilon'')t \leq |A_{ij}| \leq (1 + 2\epsilon'')t$  for all  $i$  and  $j$ . In order to make  $\bigcup_{j=1}^{k-1} A_{1j}$  a partition of  $V_1$ , we move the vertices of  $V_1'$  to  $A_{11}$ . Since  $|V_1'| < \epsilon t$ , we still have  $||A_{ij}| - t| \leq 2k\epsilon''t$  after moving these vertices. On the other hand, by (7), (11), and Claim 14, we have  $d(A_{ij}, A_{i'j}) \leq 6\epsilon'' \leq 2k\epsilon''$  for  $i \neq i'$  and all  $j$  (we now have  $d(A_{11}, A_{i1}) \leq 2\epsilon''$  for all  $i \geq 2$  because  $|A_{11}|$  becomes slightly larger). Therefore  $H$  is  $(2k\epsilon'', 2k\epsilon'')$ -approximate to  $\Theta_{k \times (k-1)}(t)$ . By the definitions of  $\epsilon''$  and  $\epsilon'$ ,

$$2k\epsilon'' = 4k^2\sqrt{\epsilon'} = 4k^2\sqrt{16(k-1)^4(3\epsilon)^{1/2k-3}} \leq 16k^4\epsilon^{1/2k-2},$$

where the last inequality is equivalent to  $(\frac{k-1}{k})^2 3^{1/2k-2} \leq 1$  or  $3^{1/2k-1} \leq \frac{k}{k-1}$ , which holds because  $3 \leq 1 + \frac{2^{k-1}}{k-1} \leq (1 + \frac{1}{k-1})^{2^{k-1}}$  for  $k \geq 2$ .

This completes the proof of Lemma 9.  $\square$

We are ready to prove Lemma 7.

*Proof of Lemma 7.* First assume that  $G \in \mathcal{G}_3(n)$  is minimal, namely,  $G$  satisfies the minimum partite degree condition but removing any edge of  $G$  will destroy this condition. Note that this assumption is only needed by Claim 20.

Given  $0 < \Delta \leq 1$ , let

$$\alpha = \frac{1}{2k} \left( \frac{\Delta}{24k(k-1)\sqrt{2k}} \right)^{2^{k-1}}. \quad (15)$$

Without loss of generality, assume that  $x, y \in V_1$  and  $y$  is *not* reachable by  $\alpha^3 n^{k-1}$   $(k-1)$ -sets or  $\alpha^3 n^{2k-1}$   $(2k-1)$ -sets from  $x$ .

For  $2 \leq i \leq k$ , define

$$\begin{aligned} A_{i1} &= V_i \cap (N(x) \setminus N(y)), & A_{ik} &= V_i \cap (N(y) \setminus N(x)), \\ B_i &= V_i \cap (N(x) \cap N(y)), & A_{i0} &= V_i \setminus (N(x) \cup N(y)). \end{aligned}$$

Let  $B = \bigcup_{i \geq 2} B_i$ . If there are at least  $\alpha^3 n^{k-1}$  copies of  $K_{k-1}$  in  $B$ , then  $x$  is reachable from  $y$  by at least  $\alpha^3 n^{k-1}$   $(k-1)$ -sets. We thus assume there are less than  $\alpha^3 n^{k-1}$  copies of  $K_{k-1}$  in  $B$ .

Clearly, for  $i \geq 2$ ,  $A_{i1}$ ,  $A_{ik}$ ,  $B_i$  and  $A_{i0}$  are pairwise disjoint. The following claim bounds the sizes of  $A_{ik}$ ,  $B_i$  and  $A_{i0}$ .

**Claim 15.** (1)  $(1 - k^2\alpha)\frac{n}{k} < |A_{i1}|, |A_{ik}| \leq (1 + k\alpha)\frac{n}{k}$ ,  
 (2)  $(k - 2 - 2k\alpha)\frac{n}{k} \leq |B_i| < (k - 2 + k(k-1)\alpha)\frac{n}{k}$ ,  
 (3)  $|A_{i0}| < (k+1)\alpha n$ .

*Proof.* For  $v \in V$ , and  $i \in [k]$ , write  $N_i(v) := N(v) \cap V_i$ . By the minimum degree condition, we have  $|A_{i1}|, |A_{ik}| \leq (1/k + \alpha)n$ . Since  $N_i(x) = A_{i1} \cup B_i$ , it follows that

$$|B_i| \geq \left(\frac{k-1}{k} - \alpha\right)n - \left(\frac{1}{k} + \alpha\right)n. \quad (16)$$

If some  $B_i$ , say  $B_k$ , has at least  $(\frac{k-2}{k} + (k-1)\alpha)n$  vertices, then there are at least  $\prod_{i=2}^k |B_i| - (i-2)(\frac{1}{k} + \alpha)n$  copies of  $K_{k-1}$  in  $B$ . By (16) and  $|B_k| \geq (\frac{k-2}{k} + (k-1)\alpha)n$ , this is at least

$$\begin{aligned} & \alpha n \cdot \prod_{i=2}^{k-1} \left( \frac{k-1}{k} - \alpha \right) n - (i-1) \left( \frac{1}{k} + \alpha \right) n \\ &= \alpha n \cdot \prod_{i=2}^{k-1} \left( \frac{k-i}{k} - i\alpha \right) n \\ &\geq \alpha n \cdot \prod_{i=2}^{k-1} \left( \frac{k-i-\frac{1}{2}}{k} \right) n \quad \text{because } 2k^2\alpha \leq 1, \\ &\geq \alpha n \cdot \frac{1}{2} \left( \frac{n}{k} \right)^{k-2} \\ &\geq \alpha^2 n^{k-1} \quad \text{because } 2k^{k-2}\alpha \leq 1. \end{aligned}$$

This is a contradiction.

We may thus assume that  $|B_i| < (\frac{k-2}{k} + (k-1)\alpha)n$  for  $2 \leq i \leq k$ , as required for Part (2). As  $N_i(x) = A_{i1} \cup B_i$ , it follows that

$$|A_{i1}| > \left(\frac{k-1}{k} - \alpha\right)n - \left(\frac{k-2}{k} + (k-1)\alpha\right)n = \left(\frac{1}{k} - k\alpha\right)n.$$

The same holds for  $|A_{ik}|$  thus Part (1) follows. Finally

$$|A_{i0}| = |V_i| - |N_i(x)| - |A_{ik}| < n - \left(\frac{k-1}{k} - \alpha\right)n - \left(\frac{1}{k} - k\alpha\right)n = (k+1)\alpha n,$$

as required for Part (3).  $\square$

Let  $t = n/k$  and  $\epsilon = 2k\alpha$ . By the minimum degree condition, every vertex  $u \in B$  is nonadjacent to at most  $(1 + k\alpha)n/k < (1 + \epsilon)t$  vertices in other color classes of  $B$ . By Claim 15,  $|B_i| \geq (k-2-2k\alpha)\frac{n}{k} = (k-2-\epsilon)t \geq (k-2)(1-\epsilon)t$ . Thus  $G[B]$  is a  $(k-1)$ -partite graph that satisfies the assumptions of Lemma 9. We assumed that  $B$  contains less than  $\alpha^3 n^{k-1} < \epsilon^2 t^{k-1}$  copies of  $K_{k-1}$ , so by Lemma 9,  $B$  is  $(\alpha', \alpha')$ -approximate to  $\Theta_{(k-1) \times (k-2)}(\frac{n}{k})$ , where

$$\alpha' := 16(k-1)^4(2k\alpha)^{1/2^{k-3}}.$$

This means that we can partition  $B_i$ ,  $2 \leq i \leq k$ , into  $A_{i2} \cup \dots \cup A_{i(k-1)}$  such that  $(1 - \alpha')\frac{n}{k} \leq |A_{ij}| \leq (1 + \alpha')\frac{n}{k}$  for  $2 \leq j \leq k-1$  and

$$\forall 2 \leq i < i' \leq k, 2 \leq j \leq k-1, \quad d(A_{ij}, A_{i'j}) \leq \alpha'. \quad (17)$$

Together with Claim 15 Part (1), we obtain that (using  $k^2\alpha \leq \alpha'$ )

$$\forall 2 \leq i \leq k, 1 \leq j \leq k, \quad (1 - \alpha')\frac{n}{k} \leq |A_{ij}| \leq (1 + \alpha')\frac{n}{k}. \quad (18)$$

Let  $A_{ij}^c = V_i \setminus A_{ij}$  denote the complement of  $A_{ij}$ . The following claim is an analog of Claim 10, and its proof is almost the same – after we replace  $(1 + \epsilon)t$  with  $(1 + k\alpha)n/k$  and  $\epsilon'$  with  $\alpha'$  (and we use  $\alpha \ll \alpha'$ ). We thus omit the proof.

**Claim 16.** *Let  $2 \leq i \neq i' \leq k$ ,  $1 \leq j \neq j' \leq k$ , and  $\{j, j'\} \neq \{1, k\}$ .*

- (1)  $d(A_{ij}, A_{i'j'}) \geq 1 - 3\alpha'$  and  $d(A_{ij}, A_{i'j}^c) \geq 1 - 3\alpha'$ .
- (2) *All but at most  $\sqrt{3\alpha'}$  vertices in  $A_{ij}$  are  $\sqrt{3\alpha'}$ -typical to  $A_{i'j'}$ ; at most  $\sqrt{3\alpha'}$  vertices in  $A_{ij}$  are  $\sqrt{3\alpha'}$ -typical to  $A_{i'j}^c$ .*  $\square$

Now let us study the structure of  $V_1$ . Let  $\alpha'' = 2k\sqrt{\alpha'}$ . Recall that  $N(xv) = N(x) \cap N(v)$ . Let  $V'_1$  be the set of the vertices  $v \in V_1$  such that there are at least  $\alpha n^{k-1}$  copies of  $K_{k-1}$  in each of  $N(xv)$  and  $N(yv)$ . We claim that  $|V'_1| < 2\alpha n$ . Otherwise, since a  $(k-1)$ -set intersects at most  $(k-1)n^{k-2}$  other  $(k-1)$ -sets, there are at least

$$2\alpha n \cdot \alpha n^{k-1} (\alpha n^{k-1} - (k-1)n^{k-2}) > \alpha^3 n^{2k-1}$$

copies of  $(2k-1)$ -sets connecting  $x$  and  $y$ , a contradiction.

Let  $\tilde{V}_1 := V_1 \setminus V'_1$ . The following claim is an analog of Claim 12 for Lemma 9 and can be proved similarly. The only difference between their proofs is that here we find at least  $\alpha n^{k-1}$  copies of  $K_{k-1}$  in each of  $N(xv)$  and  $N(yv)$ , which contradicts the definition of  $\tilde{V}_1$ .

**Claim 17.** *Given  $v \in \tilde{V}_1$  and  $2 \leq i \leq k$ , there exists  $j \in [k]$  such that  $|N_{A_{ij}}(v)| < \alpha'' t$ .*  $\square$

Fix an vertex  $v \in \tilde{V}_1$ . Claim 17 implies that for each  $2 \leq i \leq k$ , there exists  $\ell_i$  such that  $|N_{A_{i\ell_i}}(v)| < \alpha'' t$ . Our next claim is an analog of Claim 13 for Lemma 9 and can be proved similarly.

**Claim 18.** *We have  $\ell_2 = \ell_3 = \dots = \ell_k$ .*  $\square$

We now define  $A_{1j} := \{v \in \tilde{V}_1 : |N_{A_{2j}}(v)| < \alpha'' t\}$  for  $j \in [k]$ . By Claims 17 and 18, this yields a partition of  $\tilde{V}_1 = \bigcup_{j=1}^k A_{1j}$  such that

$$d(A_{1j}, A_{ij}) < \frac{\alpha'' t |A_{1j}|}{|A_{1j}| |A_{ij}|} \leq \frac{\alpha'' t}{(1 - \alpha') t} < (1 + 2\alpha') \alpha'' \text{ for } i \geq 2 \text{ and } j \geq 1. \quad (19)$$

For  $v \in A_{1j}$ , we have  $|N_{A_{ij}}(v)| < \alpha'' t$  for  $i \geq 2$ . By the minimum degree condition and (18),

$$|A_{ij}^c \setminus N(v)| \leq (\frac{1}{k} + \alpha) n - (|A_{ij}| - \alpha'' t) < 2\alpha'' t. \quad (20)$$

By (18) and (20), we derive that

$$\bar{d}(A_{1j}, A_{ij'}) < \frac{|A_{1j}| \cdot 2\alpha'' t}{|A_{1j}| |A_{ij'}|} \leq \frac{2\alpha'' t}{(1 - \alpha') t} < 3\alpha'' \text{ for } i \geq 2 \text{ and } j \neq j'. \quad (21)$$

We claim that  $|A_{1j}| \leq (1 + \alpha) t + (1 + 2\alpha') \alpha'' |A_{1j}|$  for all  $j$ . Otherwise, by the minimum degree condition, we have  $\deg_{A_{1j}}(v) > (1 + 2\alpha') \alpha'' |A_{1j}|$  for all  $v \in A_{1j}$ , and consequently  $d(A_{1j}, A_{ij}) > (1 + 2\alpha') \alpha''$ , contradicting (19). We thus conclude that

$$|A_{1j}| \leq \frac{1 + \alpha}{1 - (1 + 2\alpha') \alpha''} t < (1 + 2\alpha'') \frac{n}{k}. \quad (22)$$

Since  $|V'_1| \leq 2\alpha n$ , we have  $|\bigcup_{j=1}^k A_{1j}| = |V_1 \setminus V'_1| \geq |V_1| - 2\alpha n$ . Using (22), we now obtain a lower bound for  $|A_{1j}|$ ,  $j \in [k]$ .

$$|A_{1j}| \geq n - (k-1)(1 + 2\alpha'') \frac{n}{k} - 2\alpha n \geq (1 - 2k\alpha'') \frac{n}{k}. \quad (23)$$

It remains to show that  $d(A_{i1}, A_{i'1})$  and  $d(A_{ik}, A_{i'k})$ ,  $2 \leq i, i' \leq k$ , are small. First we show that if both densities are reasonably large then there are too many reachable  $(2k-1)$ -sets from  $x$  to  $y$ . The proof resembles the one of Claim 14.

**Claim 19.** *For  $2 \leq i \neq i' \leq k$ , either  $d(A_{i1}, A_{i'1}) \leq 6\alpha''$  or  $d(A_{ik}, A_{i'k}) \leq 6\alpha''$ .*

*Proof.* Suppose instead, that say  $d(A_{(k-1)1}, A_{k1}), d(A_{(k-1)k}, A_{kk}) > 6\alpha''$ . Fix a vertex  $v_1$  in  $A_{1j}$ , for some  $2 \leq j \leq k-1$ , say  $v_1 \in A_{12}$ . We construct two vertex disjoint copies of  $K_{k-1}$  in  $N(xv_1)$  and  $N(yv_1)$  as follows. We first select  $k-3$  sets  $A_{ij}$  with  $2 \leq i \leq k-2$  and  $3 \leq j \leq k-1$  such that no two of them are on the same row or column – there are  $(k-3)!$  choices. Fix one of them, say  $A_{23}, \dots, A_{(k-2)(k-1)}$ . For  $2 \leq i \leq k-2$ , we select  $v_i \in N_{A_{i(i+1)}}(v_1 \dots v_{i-1})$  that is  $\sqrt{3\alpha'}$ -typical to  $A_{(k-1)1}, A_{k1}$  and  $A_{j(j+1)}$ ,  $i < j \leq k-2$ . By Claim 16 and (20), there are at least

$$(1 - (k-2)\sqrt{3\alpha'}) |A_{i(i+1)}| - (k\alpha + \alpha' + \alpha'') \frac{n}{k} \geq \frac{n}{2k}$$

such  $v_i$ . After selecting  $v_2, \dots, v_{k-2}$ , we select two adjacent vertices  $v_{k-1} \in A_{(k-1)1}$  and  $v_k \in A_{k1}$  such that  $v_{k-1}$  and  $v_k$  are in  $N(v_1 \cdots v_{k-2})$ . For  $j = k-1, k$ , we know that  $N(v_1)$  misses at most  $(k\alpha + \alpha' + \alpha'')n/k$  vertices in  $A_{j1}$  and at most  $(k-3)\sqrt{3\alpha'}|A_{j1}|$  vertices of  $A_{j1}$  are not in  $N(v_2 \cdots v_{k-2})$ . Since  $d(A_{(k-1)1}, A_{k1}) > 6\alpha''$ , there are at least

$$6\alpha''|A_{(k-1)1}||A_{k1}| - (k\alpha + \alpha' + \alpha'')\frac{n}{k}(|A_{(k-1)1}| + |A_{k1}|) \\ - 2(k-3)\sqrt{3\alpha'}|A_{(k-1)1}||A_{k1}| \geq 6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2$$

such pairs  $v_{k-1}, v_k$ . Hence  $N(xv_1)$  contains at least

$$(k-3)! \left(\frac{n}{2k}\right)^{k-3} 6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2 \geq \sqrt{\alpha'}\left(\frac{n}{k}\right)^{k-1} \geq \sqrt{\alpha n^{k-1}}$$

copies of  $K_{k-1}$ . Let  $C$  be such a copy of  $K_{k-1}$ . Then we follow the same procedure and construct a copy of  $K_{k-1}$  on  $N(yv_1) \setminus C$ . After fixing  $k-3$  sets  $A_{ij}$  with  $2 \leq i \leq k-2$  and  $3 \leq j \leq k-1$  such that no two of them are on the same row or column, there are still at least  $\frac{n}{2k}$  such  $v_i$  for  $2 \leq i \leq k-2$ . Then, as  $d(A_{ik}, A_{i'k}) > 6\alpha''$ , there are at least  $6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2$  choices of  $v_{k-1} \in A_{(k-1)k}$  and  $v_k \in A_{kk}$  such that  $v_{k-1}$  and  $v_k$  are in  $N(v_1 \cdots v_{k-2})$ . This gives at least  $\sqrt{\alpha n^{k-1}}$  copies of  $K_{k-1}$  in  $N(yv_1)$ . Then, since there are at least  $|V_1| - |A_{11}| - |A_{1k}| \geq \alpha n$  choices of  $v_1$ , totally there are at least  $\alpha n(\sqrt{\alpha n^{k-1}})^2 = \alpha^2 n^{2k-1}$  reachable  $(2k-1)$ -sets from  $x$  to  $y$ , a contradiction.  $\square$

Next we show that if any of  $d(A_{i1}, A_{i'1})$  or  $d(A_{ik}, A_{i'k})$ ,  $2 \leq i, i' \leq k$ , is sufficiently large, then we can remove edges from  $G$  such that the resulting graph still satisfies the minimum degree condition, which contradicts the assumption that  $G$  is minimal.

**Claim 20.** For  $2 \leq i \neq i' \leq k$ ,  $d(A_{i1}, A_{i'1}), d(A_{ik}, A_{i'k}) \leq 6k\sqrt{\alpha''}$ .

*Proof.* Without loss of generality, assume that  $d(A_{2k}, A_{3k}) > 6k\sqrt{\alpha''}$ . By Claim 19, we have  $d(A_{21}, A_{31}) < 6\alpha''$ . Combining this with (17), we have  $d(A_{2j}, A_{3j}) < 6\alpha''$  for all  $j \in [k-1]$ . Now fix  $j \in [k-1]$ . The number of non-edges between  $A_{2j}$  and  $A_{3j}$  satisfies  $\bar{e}(A_{2j}, A_{3j}) > (1-6\alpha'')|A_{2j}||A_{3j}|$ . By the minimum degree condition and (18),

$$\bar{e}(A_{2k}, A_{3j}) < (1 + k\alpha)\frac{n}{k}|A_{3j}| - (1-6\alpha'')|A_{2j}||A_{3j}| \leq 7\alpha''\frac{n}{k}|A_{3j}|.$$

Using (18) again, we obtain that

$$d(A_{2k}, A_{3j}) \geq 1 - \frac{7\alpha''\frac{n}{k}|A_{3j}|}{|A_{2k}||A_{3j}|} \geq 1 - 8\alpha''.$$

This implies that  $d(A_{2k}, A_{3k}^c) \geq 1-8\alpha''$ . Similarly we derive that  $d(A_{3k}, A_{2k}^c) \geq 1-8\alpha''$ . For  $i = 2, 3$ , define  $A_{ik}^T$  as the set of the vertices in  $A_{ik}$  that are  $\sqrt{8\alpha''}$ -typical to  $A_{(5-i)k}^c$ . Thus  $|A_{ik} \setminus A_{ik}^T| \leq \sqrt{8\alpha''}|A_{ik}|$ .

Let  $A_{ik}^{T_1} = \{v \in A_{ik}^T : \deg_{A_{(5-i)k}^c}(v) \leq \sqrt{8\alpha''}|A_{jk}^c|\}$  and  $A_{ik}^{T_2} = A_{ik}^T \setminus A_{ik}^{T_1}$ . For  $u \in A_{2k}^{T_2}$ , we have

$$\deg_{V_3}(u) = \deg_{A_{3k}^c}(u) + \deg_{A_{3k}}(u) > (1 - \sqrt{8\alpha''})|A_{3k}^c| + \sqrt{8\alpha''}|A_{3k}^c| = |A_{3k}^c|.$$

Since  $|A_{3k}^c| \geq \deg_{V_3}(x)$  and  $|A_{3k}^c|$  is an integer, we conclude that  $\deg_{V_3}(u) \geq \deg_{V_3}(x) + 1$ . Similarly we can derive that  $\deg_{V_2}(v) \geq \deg_{V_2}(x) + 1$  for every  $v \in A_{3k}^{T_2}$ . If there is an edge  $uv$  joining some  $u \in A_{2k}^{T_2}$  and some  $v \in A_{3k}^{T_2}$ , then we can delete this edge and the resulting graph still satisfies the minimum degree condition. Therefore we may assume that there is no edge between  $A_{2k}^{T_2}$  and  $A_{3k}^{T_2}$ .

Then

$$\begin{aligned}
e(A_{2k}, A_{3k}) &\leq e(A_{2k} \setminus A_{2k}^T, A_{3k}) + e(A_{2k}, A_{3k} \setminus A_{3k}^T) + e(A_{2k}^{T_1}, A_{3k}^{T_1}) + e(A_{2k}^T, A_{3k}^{T_1}) \\
&\leq 2\sqrt{8\alpha''}|A_{2k}||A_{3k}| + |A_{2k}^{T_1}|\sqrt{8\alpha''}|A_{3k}^c| + |A_{3k}^{T_1}|\sqrt{8\alpha''}|A_{2k}^c| \\
&\leq \sqrt{8\alpha''}(2|A_{2k}||A_{3k}| + |A_{2k}||A_{3k}^c| + |A_{3k}||A_{2k}^c|) \\
&= \sqrt{8\alpha''}(|A_{2k}||V_3| + |A_{3k}||V_2|) \\
&\leq 3\sqrt{\alpha''} \cdot 2k|A_{2k}||A_{3k}| \quad \text{by (18)}.
\end{aligned}$$

Therefore  $d(A_{2k}, A_{3k}) \leq 6k\sqrt{\alpha''}$ .  $\square$

In summary, by (18), (22) and (23), we have  $(1 - 2k\alpha'')^{\frac{n}{k}} \leq |A_{ij}| \leq (1 + 2\alpha'')^{\frac{n}{k}}$  for all  $i$  and  $j$ . In order to make  $\bigcup_{j=1}^k A_{ij}$  a partition of  $V_i$ , we move the vertices of  $V_1'$  to  $A_{11}$  and the vertices of  $A_{i0}$  to  $A_{i2}$  for  $2 \leq i \leq k$ . Since  $|V_1'| < 2\alpha n$  and  $|A_{i0}| \leq (k+1)\alpha n$ , we have  $||A_{ij}| - \frac{n}{k}| \leq 2k\alpha''^{\frac{n}{k}}$  after moving these vertices. On the other hand, by (17), (19), and Claim 20, we have  $d(A_{ij}, A_{i'j}) \leq 6k\sqrt{\alpha''}$  for  $i \neq i'$  and all  $j$ . (In fact, for  $i \geq 2$ , we now have  $d(A_{11}, A_{i1}) \leq 2\alpha''$  as we added at most  $2\alpha n$  vertices to  $A_{11}$ . For  $i' > i \geq 2$ , we now have  $d(A_{12}, A_{i2}) \leq 2\alpha''$  and  $d(A_{i2}, A_{i'2}) \leq 2\alpha'$  as we moved at most  $(k+1)\alpha n$  vertices to  $A_{i2}$ .) Therefore after deleting edges,  $G$  is  $(2k\alpha'', 6k\sqrt{\alpha''})$ -approximate to  $\Theta_{k \times k}(n/k)$ . By (15), and the definitions of  $\alpha''$  and  $\alpha'$ ,  $G$  is  $(\Delta/6, \Delta/2)$ -approximate to  $\Theta_{k \times k}(n/k)$ .  $\square$

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